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Epsilon Dominance and Constraint Partitioning in Multiple Objective Problems

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Abstract. In this paper we consider efficient sets of multiple objective problems, in which the feasible action set is the intersection of two other sets, and where one of these sets has a special structure, such as an assignment or transportation structure. The objective is to find the efficient set of the special structure set, and its intersection with the other set, and to examine how good an approximation this set is to the desired efficient set. The approximation set is called an ϵ-efficient solution set. Some theoretical partition results are given for a special constraint structure with upper bounds on the objective function levels. For the case of 0-efficient solution sets, and finite explicit sets, a computational cost analysis of two computational sequences is given. We also consider two other 0-efficient solution set cases. Then ϵ-efficiency is considered for linear problems. Finally, the approach is illustrated by a special multiple objective transportation problem.

Key words: Multiple objectives, Vector optimization, Constraints

1. Introduction

Many real life problems involve evaluating a set of potential actions in terms of several measures of performance such as cost and time. These give rise to what are known as multiple objective problems (see, e.g., [1–5]). Let us suppose that we have K such measures of performance, giving rise to a performance vector $f(x) = (f^1(x), f^2(x), \cdot, \cdot, f^k(x), \cdot, \cdot, f^K(x))$ for each potential action x in a set X. In the ideal situation there will be a known preference function Ψ such that, if $x \in X$ and $y \in X$, then x is at least as good as y if and only if $\Psi(f(x)) \leqslant \Psi(f(y))$.

In such cases the decision problem becomes

$$\min_{x \in X}[\Psi(f(x))].$$

Note that we have cast our problem in minimisation form, but it might equally well be cast in maximisation form.

In general, it is difficult to find explicitly, at least initially, the function Ψ, and it leads to trying to find some useful property of Ψ which can be used to generate some actions among which an optimal one, in terms of the implicit function Ψ, lies. The weakest property of Ψ is a monotone one. For $f \in R^K, g \in R^K$ let us define $f \leqslant g$ if and only if $f_k \leqslant g_k, 1 \leqslant k \leqslant K$. Then the monotone property is that, if $x \in X$ and $y \in X$, and $f(x) \leqslant f(y)$, then

$$\Psi(f(x)) \leqslant \Psi(f(y)),$$

and if $f(x) \leqslant f(y), f(x) \neq f(y)$, then

$$\Psi(f(x)) < \Psi(f(y)).$$

In such cases, if $x \in X$ and $y \in X$, with $f(x) \leqslant f(y), f(x) \neq f(y)$, then the vector $f(x)$ is said to dominate the vector $f(y)$, and the solution x is said to dominate the solution y, and y can be eliminated without loss. The set of non-dominated $x \in X$ is called the efficient solution set, X^*, of X with respect to $\{f(\cdot), \leqslant\}$.

If $f(X^*)$ is the image of X^* in the objective function space, then the optimisation problem reduces to

$$\min_{f \in f(X^*)} [\psi(f)].$$

References [6] and [7] attempt to characterize $f(X^*)$ as a first step towards solving the above problem. The set $f(X^*)$ is, in general, even for linear objective functions and polyhedral X, a non-convex set. Hence, any insights into the characterization of $f(X^*)$ can be of assistance in solving the given optimisation problem.

A more general form of the problem of this paper has been studied, in which the optimisation problem takes the form

$$\min_{x \in X^*} [\tau(x)],$$

where $\tau(\cdot)$ is convex, or specifically linear, and X is convex, or specifically polyhedral, or polytopal (see, e.g., [8–12]). This problem is somewhat harder than that of this paper, but, nontheless, the ability to find, or characterize, $f(X^*)$, will be of use in these situations.

However, finding X^* and $f(X^*)$ can be much more difficult than finding an optimal solution to a scalar problem, such as minimising $f^k(x)$ over $x \in X$ for some k. Now, even in scalar optimisation problems, if the computational load is high, finding an optimal solution to the scalar problem may be replaced by finding a solution whose objective function value is within a prespecified positive distance ϵ of the optimal value. This philosophy clearly becomes of more importance in the multiple objective efficient solution set case. In addition, because of the vastly increased computational load, higher values of ϵ may be acceptable. In addition to this, short cuts which use special aspects of the structure of the problem can be valuable.

In this paper, we generalise the notion of an ϵ-optimal solution set for a scalar problem to the notion of an ϵ-efficient solution set, and use a class of problems with a specific structure to try to short cut the computations. We will consider a class of problems where X takes the form

$$X = Z \cap Y \tag{1}$$

where Z and Y are individually simpler than X, and we assume that X is compact in R_+^n and $f(\cdot)$ is continuous on R_+^n.

In addition to this, we will extend the problem beyond finding X^* to a problem of finding an ϵ-dominating substitute $X^*(\epsilon)$, when $\epsilon \in R_+^K$ is specified and ϵ-dominance is defined as follows, viz. given two sets of actions U, V, with $V \subseteq U$, and $f(\cdot)$ is defined on V and U, then V is said to ϵ-dominate U with respect to $\{f(\cdot), \leqslant\}$ if

$$u \in U \to \exists v \in V \text{ with } f(v) - \epsilon \leqslant f(u). \tag{2}$$

If ϵ is acceptably small, and if V^* is the efficient set of V with respect to $\{f(\cdot), \leqslant\}$ then, given the monotonic preference assumption, V^* will be almost as good as U for decision making purposes. If it is relatively easy to determine, in comparison with determining the efficient solution set, then it may be more acceptable to find V^*, even though some loss of value may arise in the final decision making process as a result.

If U^* is the efficient set of U with respect to $\{f(\cdot), \leqslant\}$, then V will be said to be an ϵ-efficient solution set for U if $V \subset U^*$ and V satisfies (2).

In our case we will set

$$U = X, \quad V = Z \cap Y^*. \tag{3}$$

It is easily demonstrated that if $\{U, V, X\}$ satisfy (1) and (3) then (see Lemma 1)

$$V = V^* \subseteq U^*. \tag{4}$$

Then, if $\{U, V, X\}$ satisfy (1)–(3), V is an ϵ-efficient solution set for U. Then we may set $X^*(\epsilon) = V$.

In Section 2 we will look at some sufficient conditions for $Z \cap Y^*$ to be the efficient solution set and to be a 0-efficient solution set. We will look at the question of preferred computational sequence and consider some special cases. In Section 3 we will examine the problem of evaluating a specific ϵ for $Z \cap Y^*$ to be an ϵ-efficient solution set for X in the linear case, specifically for $K = 2$, although the approach is generalisable to general K. We also look at a special application. We then give a summary and comments. We present no formal algorithm. The paper is intended as an exploratory one, developing a framework on the basis of which subsequent algorithms may be designed.

In what follows, if U is any set, with $f(\cdot)$ defined on U, we define

$$f(U) = \{\theta \in R^K : \theta = f(u) \text{ for some } u \text{ in } U\}.$$

Then, if V is a 0-efficient set for U, we have

$$f(V) = f(U^*).$$

Also, V is a 0-efficient set for U^* if and only if

$$f(V) = f(U^*).$$

As a result of this, 0-efficiency, and ϵ-efficiency, will be subsequently examined in the objective function space. We also note, for future use, that

$$Z \cap Y^* = X \cap Y^*.$$

2. 0-efficiency

2.1. A PRELIMINARY RESULT

From now on we will assume that the sets $\{U, V, X, Y, Z\}$ are all in R^n_+, and that $f(\cdot)$ is defined on all these sets, taking real values on these sets.

We first of all prove the following theorem.

LEMMA 1. *If $\{U, V, X\}$ satisfy (1) and (3), then (4) holds.*

Proof. Clearly, $V^* \subseteq V$. Now suppose $u \in V \backslash V^*$. Then, there exists a $v \in V$ such that $f(v) \leqslant f(u)$, $f(u) \neq f(v)$. Now $u \in V$ and $v \in V$. Hence, $u \in Y^*$ and $v \in Y^*$. This contradicts the statement that $f(v) \leqslant f(u)$, $f(u) \neq f(v)$. Hence, $V^* = V$.

Let $u \in V \backslash U^*$. Then $u \in U$ and there exists a $v \in U$ with $f(v) \leqslant f(u)$, $f(u) \neq f(v)$. Now $u \in Y^*$ and $v \in Y$, and this contradicts the statement that $f(v) \leqslant f(u)$ and $f(u) \neq f(v)$. □

THEOREM 1. *Let $a \in R^k$ be given, (1) holds and*

$$Z = \{x \in R^n_+ : f(x) \leqslant a\}. \quad (5)$$

Then

(i) $\quad X^* = Z \cap Y^*$;

(ii) $\quad Z \cap Y^*$ *is a 0-efficient solution set for* X. (6)

Proof. (i) The fact that $Z \cap Y^* \subseteq X^*$ comes from Lemma 1.

Now let $x \in X^* \backslash (Z \cap Y^*)$. Then, $x \in Y, x \in Z$ but $x \neq Y^*$. Thus, there exists a $y \in Y$ with $f(y) \leqslant f(x), f(y) \neq f(x)$. Because $x \in Z$, we have $f(x) \leqslant a$. Hence, $f(y) \leqslant a$, and thus $y \in Z$. We then have $y \in X$, $f(y) \leqslant f(x)$, $f(y) \neq f(x)$, contradicting $x \in X^*$. Thus $X^* \subseteq Z \cap Y^*$.

(ii) This comes from part (i) and Theorem 6, Chapter 2 of Reference [4]. This requires that, for all $x \in X$, the following set

$$S(x) = \{y \in X : f(y) \leqslant f(x)\}$$

is compact, and this holds for this paper because X is assumed to be compact, and $f(\cdot)$ is assumed to be continuous on R^n_+. □

This particular form of Z is used in Reference [13] for a constrained multiple objective routing problem. In the case of Theorem 2, identity (6) holds. It is possible, in other cases, for $Z \cap Y^*$ to be a 0-efficient solution set for X, but (6) not to hold.

2.2. CHOICE OF COMPUTATIONAL SEQUENCE

When $Z \cap Y^*$ is a 0-efficient solution set for X, we have $f(X^*) = f(Z \cap Y^*)$ and there is the question of how we might use this knowledge to assist the determination of $f(X^*)$. In some cases it is useful, and in other cases it is not useful. Two ways of proceeding are as follows, viz.

(a) find $f(Z \cap Y)$ and then $f(Z \cap Y^*)$;
(b) find $f(X^*)$ and then $f(Z) \cap f(Y^*)$.

We consider three classes of situation.

A. Explicit finite lists of options. Let us suppose that $\{X, Y, f(X), f(Y)\}$ are finite explicitly listed sets and that $f(X^*)$ and $f(Y^*)$ are found by some comparison procedure which eliminates non-efficient solutions. Let X and Y contain p and q members respectively. The simple way to find $f(X^*)$ and $f(Y^*)$ directly is by paired comparisons. This is computationally inefficient, but gives comparable computational complexity with other methods, and allows us a little more precision in computational time estimates (the method of Reference [14] is better). For our method, there are $p(p-1)/2$ and $q(q-1)/2$ comparisons, respectively, for finding $f(X^*)$ and $f(Y^*)$.

Let us now suppose that, on average, for any finite set W with r members, W^* contains αr members, with $0 < \alpha \leqslant 1$, and that, on average, βq members of Y are in Z, for some $0 \leqslant \beta \leqslant 1$.

Finally, let c be the cost of checking whether or not a specific x is in Z, and d be the paired comparison cost in determining the efficient solution set.

Then, the costs of finding $f(X^*)$ are

$$\text{via (a):} \quad cq + d(\beta q(\beta q - 1)/2)$$

and

$$\text{via (b):} \quad \alpha cq + d(q(q-1)/2).$$

Thus, finding $f(X^*)$ via (a) will be at least as good as by method (b) if and only if

$$c/d \leqslant q(q(1+\beta) - 1)(1-\beta)/2(1-\alpha). \quad (7)$$

For some problems (see, e.g., [13], which uses a vector minimum dynamic programming approach, similar to the approaches of Reference [4, p. 165], to find efficient solutions for a constrained routing problem) Z is given by (5), x corresponds to a route, and Y is a set of routes.

In this case, checking whether or not $x \in Z$ is computationally equivalent to a paired comparison in the efficiency procedure. Thus, $c = d$. In general, for large q, β is likely to be close to 0 and inequality (7) approaches

$$q(q-1) \geqslant 2(1-\alpha),$$

and is almost certain to be true. Thus, for this class of problem and others, it is likely that method (a) is preferable. Nontheless, it is conceivable that, for some problems in this class, method (b) is preferable.

B. Implicit options defined by constraints for real variables. Let us suppose that (1) holds and that Z and Y are implicitly defined by the following constraints with $a \in R^K$, $b \in R^m$, $F \in R^{K \times n}$, $B \in R^{m \times n}$, viz.

$$Z = \{x \in R_+^n : Fx \leqslant a\}, \quad Y = \{x \in R_+^n : Bx \leqslant b\} \tag{8}$$

and

$$f(x) = Fx, F \geqslant 0. \tag{9}$$

Now it may be easier to find $f(Y^*)$ by some method than to find $f(X^*)$ by the same method. We will expand upon this for the linear situation given by (8), (9) for $K = 2$, using the weighting vector method for determining efficient solutions.

From Reference [4, Chapter 4, Corollary 1.2]

$$Y^* = \{y \in Y : \exists \lambda \in R_+^2, \lambda > 0, \lambda_1 + \lambda_2 = 1, \text{ such that} \\ \lambda F y \leqslant \lambda F x \quad \forall x \in Y\}. \tag{10}$$

Using (10) we see that $f(Y^*)$ is a piecewise-linear connected set in R_+^2, and that $f(Y^*)$ is characterized by a set $f_E(Y^*)$ of T extreme points $\{\theta^t, 1 \leqslant t \leqslant T\} \subseteq R_+^2$, with end points θ^1, θ^T, with

$$\theta_1^t > \theta_1^{t+1}, \quad 1 \leqslant t \leqslant T-1$$

and

$$f(Y^*) = \bigcup_{t=1}^{T-1} \bigcup_{\gamma \in [0,1]} \{\theta \in R_+^2 : \theta = \gamma \theta^{t+1} + (1-\gamma)\theta^t\}. \tag{11}$$

Similarly, $f(X^*)$ is a piecewise-linear connected set in R_+^2, and $f(X^*)$ is characterized by a set $f_E(X^*)$ of S extreme points $\{\psi^s, 1 \leqslant s \leqslant S\} \subseteq R_+^2$ with end points ψ^1, ψ^2, with

$$\psi_1^s > \psi_1^{s+1}, 2 \leqslant s \leqslant S-2, \quad \psi_1^1 \geqslant \psi_1^2, \quad \psi_1^{S-1} \geqslant \psi_1^S,$$

and

$$f(X^*) = \bigcup_{s=1}^{S-1} \bigcup_{\gamma \in [0,1]} \{\theta \in R_+^2 : \theta = \gamma \psi^{t+1} + (1-\gamma)\psi^t\}. \tag{12}$$

Note that it is possible, with this presentation, to have $\psi^1 = \psi^2$ and /or $\psi^{S-1} = \psi^S$, as stated in (14) and (16) to follow.

Because (6) holds, we see that $f(X^*)$ is a connected subset of $f(Y^*)$ and that there exists a pair $\{t_1, t_2\} \subseteq \{1, 2, \cdot, \cdot, T\}$ (possibly not distinct) such that

$$\psi^s = \theta^{s+t_1-2}, \quad 2 \leqslant s \leqslant t_2 - t_1 + 2 = S - 1, \tag{13}$$

$$\psi^1 = \psi^2 = \theta^1 \text{ if } t_1 = 1, \tag{14}$$

$$\begin{aligned} \psi^1 &= \gamma_1\theta^{t_1} + (1-\gamma_1)\theta^{t_1-1} : \gamma_1 \\ &= \min[\gamma \in [0,1] : \gamma\theta_1^{t_1} + (1-\gamma)\theta^{t_1-1} \in f(X)] \text{ if } t_1 \neq 1, \end{aligned} \tag{15}$$

$$\psi^S = \psi^{S-1} = \theta^T \text{ if } t_2 = T, \tag{16}$$

$$\begin{aligned} \psi^S &= \gamma_S\theta^{t_2+1} + (1-\gamma_S)\theta^{t_2} : \gamma_S \\ &= \max[\gamma \in [0,1] : \gamma\theta^{t_2+1} + (1-\gamma)\theta^{t_2} \in f(X)] \text{ if } t_2 \neq T. \end{aligned} \tag{17}$$

In effect (13)–(17) generate $f(X^*)$ from $f(Y^*)$ by following $\{\theta^t, 1 \leqslant t \leqslant T\}$, beginning with $t = 1$, until the first θ^t is encountered such that $\theta^t = Fx$ for some $x \in X$, viz. $t = t_1$. The sequence is then reversed, beginning at $t = T$, until the first t is encountered with $\theta^t = Fx$ for some $x \in X$, viz. $t = t_2$.

If $t_1 = 1$, then $s = 1$ and $s = 2$ are identified with $t = 1$. Otherwise, for some $t_1 \neq 1$, θ^{t_1} is the first θ^t encountered with $\theta^t = Fx$ for some $x \in X$. Then, in effect, (15) simply finds that part of the line joining θ^{t_1-1} and θ^{t_1} which is generated by members of X.

A similar explanation applies for the reverse sequence.

The determination of $\{\theta^t, 1 \leqslant t \leqslant T\}$ is carried out by using parametric linear programming in λ_1, as a result of identity (10).

It is not necessary to check each θ^t to see if $\theta^t = Fx$ for some $x \in X$. If we begin with $\lambda = (0, 1)$ (removing any inefficient solutions which may arise because $\lambda \ngtr 0$), we merely check all points θ^t generated until we reach $t = t_1$. We may reverse the process, beginning with $\lambda = (1, 0)$, and only check until $t = t_2$ is found.

In general, $T > S$, and thus more extreme points arise in $f(Y^*)$ than in $f(X^*)$. However, the parametric linear programming involved in finding $f(Y^*)$ can be somewhat less than that involved in finding $f(X^*)$.

For transportation problems, for example, if Y corresponds to the unconstrained transportation problem, and Z corresponds to the additional constraints, then the special structure of the unconstrained transportation problem considerably assists the determination of $f(Y^*)$, and this structure is lost once additional constraints are added.

C. Implicit options defined by constraints for discrete variables. There are some discrete problems where special structures are useful in an optimisation process and for which $f(Y^*)$ is more readily found than is $f(X^*)$, using a specific method.

In Reference [15], a special form of assignment problem is studied, in which $f(x)$ takes the form, with $K^2 = n$,

$$f^k(x) = \sum_{i=1}^{K} c_{ik} x_{ik}, \quad 1 \leqslant k \leqslant K. \tag{18}$$

In this class of problems, i may be a job, and k may be the machine, and, for each k, (8) gives the total time for machine k for assignment x.

For this class of discrete problems, the weighting vector method still gives all of the efficient solutions. Once additional constraints are added, this result fails to hold. If Y corresponds to the standard assignment constraints, then $f(Y^*)$ may be found fairly easily, and if the additional constraints, defining Z, are such as to make $Z \cap Y^*$ a 0-efficient solution set for $X, f(Z) \cap f(Y^*)$ will give $f(X^*)$.

3. ϵ-efficiency in the linear case

3.1. ϵ-EVALUATION

In the general case, finding, for a given ϵ whether or not $Z \cap Y^*$ is an ϵ-efficient solution set for X, means showing whether or not the following is true, viz.

$$\forall x \in X, \exists y \in Z \cap Y^* : f(y) - \epsilon \leqslant f(x). \tag{19}$$

This is, in general, quite a difficult problem.

In the case when $\epsilon_k = \sigma$, $1 \leqslant k \leqslant K$, demonstrating (17) is equivalent to showing that

$$\max_{x \in X} \left[\min_{y \in Z \cap Y^*} \left[\max_{1 \leqslant k \leqslant K} \left[f^k(y) - f^k(x) \right] \right] \right] \leqslant \sigma. \tag{20}$$

Even the demonstration of (20) can be quite difficult. Finding the smallest σ for which (20) holds is even harder.

In this section, we will confine ourselves to the linear form (8), (9), with $K = 2$ but with the constraint matrix F for Z in (8) replaced by a more general constraint matrix $A \in R^{l \times n}$.

$f(Y^*)$ is given by (11). However, although $f(X^*)$ still takes the form (12), the identity with (13)–(17) does not necessarily hold.

We may, however, in principle adopt a similar approach to that of (13)–(17). Using (11) we see that $f(Z) \cap f(Y^*)$ takes the form

$$f(Z) \cap f(Y^*) = \bigcup_{t=1}^{T-1} \bigcup_{\gamma \in [\underline{\gamma}_t, \overline{\gamma}_t]} \{\theta \in R^2_+ : \theta = \gamma \theta^{t+1} + (1-\gamma)\theta^t\}$$

where, when they exist,

$$\overline{\gamma}_t = \max[\gamma \in [0,1] : \gamma\theta^{t+1} + (1-\gamma)\theta^t \in f(Z)]$$

and

$$\underline{\gamma}_t = \min[\gamma \in [0,1] : \gamma\theta^{t+1} + (1-\gamma)\theta^t \in f(Z)].$$

In effect, $\{\overline{\gamma}_t, \underline{\gamma}_t\}$ determine the extreme points of the linear section of $f(Y^*)$ between θ^t and θ^{t+1} which lies in $f(Z)$, and hence lies in $f(X^*)$. We note that neither $\overline{\gamma}_t$ nor $\underline{\gamma}_t$ need exist, in which case the corresponding line subsection is empty.

In principle, the pairs of points $\{\overline{\xi}^t, \underline{\xi}^t\}$ given by

$$\underline{\xi}^t = \underline{\gamma}_t\theta^{t+1} + (1-\underline{\gamma}_t)\theta^t, \quad 1 \leqslant t \leqslant T-1,$$

$$\overline{\xi}^t = \overline{\gamma}_t\theta^{t+1} + (1-\overline{\gamma}_t)\theta^t, \quad 1 \leqslant t \leqslant T-1,$$

when they exist, may be found by tracing through $\{\theta^t, 1 \leqslant t \leqslant T\}$, noting that if $\theta^t \in f(Z)$ and $\theta^{t+1} \in f(Z)$, then $\underline{\gamma}_t = 0, \overline{\gamma}_t = 1$, and constructing the requisite linear subsections.

When $\epsilon = 0$ is feasible in (19), we have

$$\underline{\xi}^t = \theta^t, \quad 1 \leqslant t \leqslant T-1,$$

$$\overline{\xi}^t = \theta^{t+1}, \quad 1 \leqslant t \leqslant T-1.$$

The pairs $\{\{\overline{\xi}^t, \underline{\xi}^t\}, 1 \leqslant t \leqslant T-1\}$, determine line intervals on $f(Y^*)$ which lie in $f(Z)$. The residual region of $f(Y^*)$ consists of open or half-open, line intervals of $f(Y^*)$. To avoid undue details, we will simply let $\{W^r, 1 \leqslant r \leqslant R\}$, be the closures of these intervals (we include the end points of each such interval to simplify matters, though such points may be in $f(Z)$).

For the moment, let us assume that $Z \cap Y^* \neq \phi$. Now if $x \in X$, there exists a $y \in Y^*$ with $Fy \leqslant Fx$. Hence, an upper bound on the left-hand side expression of (20) is

$$\overline{\sigma} = \max_{x \in Y^*}\left[\min_{y \in Z \cap Y^*}\left[\max_{1 \leqslant k \leqslant 2}\left[f^k(y) - f^k(x)\right]\right]\right].$$

Since, without loss, we can drop all $f \in f(X) \cap f(Y^*)$, this is the same as

$$\overline{\sigma} = \max_{1 \leqslant r \leqslant R}\left[\max_{f \in W^r}\left[\min_{g \in f(X) \cap f(Y^*)}\left[\max_{1 \leqslant k \leqslant 2}[g_k - f_k]\right]\right]\right].$$

Now let $\{\{\overline{\tau}^r, \underline{\tau}^r\}, 1 \leqslant r \leqslant R\}$ be the extreme points of $\{W^r, 1 \leqslant r \leqslant R\}$, with $\overline{\tau}_1^r \leqslant \underline{\tau}_1^r, \overline{\tau}_2^r \geqslant \underline{\tau}_2^r$.

Also, for each W^r, $1 \leqslant r \leqslant R$, let $\{\overline{\mu}^r, \underline{\mu}^r\}$ be the nearest two members of $\cup_{1\leqslant t\leqslant T-1}\{\overline{\xi}^t, \underline{\xi}^t\}$ to W^r, with $\overline{\mu}_1^r \leqslant \overline{\tau}_1^r, \overline{\mu}_2^r \geqslant \underline{\tau}_2^r, \underline{\mu}_1^r \geqslant \underline{\tau}_1^r, \underline{\mu}_2^r \leqslant \underline{\tau}_2^r$. Then, for $f \in W^r$,

$$\min_{g\in f(X)\cap f(Y^*)} \left[\max_{1\leqslant k\leqslant 2}[g_k - f_k]\right]$$

is realised at $g = \overline{\mu}^r$ and/or $g = \underline{\mu}^r$.

Letting $f \in W^r$ take the form

$$f = \gamma\overline{\tau}^r + (1-\gamma)\underline{\tau}^r, \quad \gamma \in [0,1],$$

we see that

$$\overline{\sigma} = \max_{1\leqslant r\leqslant R}\left[\max_{0\leqslant\gamma\leqslant 1}\left[\min_{g\in\{\overline{\mu}^r,\underline{\mu}^r\}}\left[\max_{1\leqslant k\leqslant 2}[g_k - (\gamma\overline{\tau}_k^r + (1-\gamma)\underline{\tau}_k^r)]\right]\right]\right]. \tag{21}$$

The size of $\overline{\sigma}$ will depend on how well $\{\{\overline{\xi}^t, \underline{\xi}^t\}, 1 \leqslant t \leqslant T-1\}$ are distributed over $f(Y^*)$.

If $Z \cap Y^* = \phi$, then we cannot use the above method. However, we can supplement $f(Z) \cap f(Y^*)$ by additional points in $f(X^*)$ as follows.

Each extreme point of $W^r, 1 \leqslant r \leqslant R$, which is not in $f(Z)$ must be in $\{\theta^t, 1 \leqslant t \leqslant T\}$, because if $\overline{\tau}^r$ (resp. $\underline{\tau}^r$) $\in \cup_{t=1}^{T-1}\{\overline{\xi}^t, \underline{\xi}^t\}$, then $\overline{\tau}^r$ (resp. $\underline{\tau}^r$) $\in f(Z)$.

So let $T^0 \subseteq \{1, 2, \cdot, \cdot, T\}$ be the set of t for which $\theta^t \notin f(Z)$, and consider the following problem P^t, $t \in T^0$.

$$\underline{P^t} \quad \min_{x\in X}\left[\max_{1\leqslant k\leqslant 2}[\max[f^k(x) - \theta_k^t, 0]]\right]. \tag{22}$$

If ϵ^t is the value of (22), and x^t solves problem P^t, then $\epsilon^t \geqslant 0$ and

$$f^k(x^t) \leqslant \theta_k^t + \epsilon^t, t \in T^0. \tag{23}$$

If x^t uniquely solves problem P^t, then $x^t \in X^*$. If x^t does not uniquely solve P^t, the following problem Q^t will produce a solution $\hat{x}^t$ which solves P^t and lies in X^* (see Theorem 7, Chapter 1 of Reference [4]).

$$\underline{Q^t} \quad \min_{x\in X, f(x)\leqslant f(x^t)}\left[\sum_{1\leqslant k\leqslant 2} f^k(x)\right].$$

The points $\{f(x^t)$ or $f(\hat{x}^t), t \in T^0\}$, are nearest points in X^*, in a specified sense implicit in (22), to $\{\theta^t, t \in T^0\}$. If $\{\epsilon^t, t \in T^0\}$, are small, then, adding these to $\{\{\overline{\mu}^r, \underline{\mu}^r\}, 1 \leqslant r \leqslant R\}$, in (21), will tend to reduce $\overline{\sigma}$. In the extreme case, when $\epsilon^t = 0$, then, from (23) we have

$$\theta^t = f(x^t)(\text{ or } f(\hat{x}^t)) \in f(X^*),$$

because x^t (or $\hat{x}^t$) $\in Y$ and $\theta^t \in f(Y^*)$.

Note that the procedure outlined above can, in principle, be applied to all $\{\theta^t, 1 \leqslant t \leqslant T\}$, in the first instance if desired. The $\{\{\overline{\xi}^t, \underline{\xi}^t\}, 1 \leqslant t \leqslant T-1\}$, analysis simply allows some of these to be determined as being in $f(X^*)$ in the first instance, in which case $\epsilon^t = 0$ for all such θ^t.

3.2. FURTHER APPROXIMATIONS

The number, T, of extreme points of $f(Y^*)$, may be quite large. Hence it may be required that $f(Y^*)$ be reduced by some selection process before the ϵ-evaluation phase is commenced.

If we select of subset of $\{\theta^t, 1 \leqslant t \leqslant T\}$, including θ^1 and θ^T in this subset, we generate a surrogate set $f(Y^{**})$ for $f(Y^*)$. A similar ϵ-efficiency analysis may be applied to $f(Y^{**})$ as was given for $f(Y^*)$.

3.3. AN ILLUSTRATION

We will consider a standard multiple objective balanced transportation problem, where

$$Y = \left\{ x \in R_+^{M\times N} : \sum_{j=1}^{N} x_{ij} = b_i, \quad 1 \leqslant i \leqslant M \right\}, \tag{24}$$

$$Z = \left\{ x \in R_+^{M\times N} : \sum_{i=1}^{M} x_{ij} = a_j, \quad 1 \leqslant j \leqslant N \right\}, \tag{25}$$

$$f^k(x) = \sum_{i=1}^{M} \sum_{j=1}^{N} f_{ij}^k x_{ij}, \quad k = 1, 2.$$

(24) and (25) have been put in equality form, although they could be put in inequality form. Also, to conform with the general framework, we have $n = MN$.

If we find $f(Y^*)$ by the weighting factor method, given $\lambda \in (0,1)$ we need to find

$$\min_{x\in Y} \left[\sum_{i=1}^{M} \sum_{j=1}^{N} (\lambda f_{ij}^1 + (1-\lambda) f_{ij}^2) x_{ij} \right]. \tag{26}$$

Because of the special form of Y we see that, in (26), we may carry out the minimisation for each value of i separately.

For $1 \leqslant i \leqslant M$, let $\{\theta^{ti}, 1 \leqslant t \leqslant T(i)\}$, be the set of efficient solutions of the set of N vectors $\{b_i f_{ij}, 1 \leqslant j \leqslant N\}$, corresponding to $\{x_{ij}, 1 \leqslant j \leqslant N\}$, ordered so that

$$\theta_1^{ti} > \theta_1^{t+1,i}, 1 \leqslant t \leqslant T(i) - 1.$$

For $1 \leqslant i \leqslant M$, the critical λ values generated in the weighting factor process are given by $\lambda^{ti} = (\theta_2^{t+1,i} - \theta_2^{ti})/((\theta_1^{ti} - \theta_1^{t+1,i}) + (\theta_2^{t+1,i} - \theta_2^{ti})), 1 \leqslant t \leqslant T(i) - 1$.

At $\lambda = \lambda^{ti}$, in the multiple objective space R_+^2, $\theta^{t+1,i}$ and θ^{ti} are equally optimal for (26), with

$$\lambda^{ti} < \lambda^{t+1,i}, \quad 1 \leqslant t \leqslant T(i) - 1.$$

The complete set of critical λ values for the overall optimisation in (26), taken over all i, is then

$$\{\lambda^{ti}, \quad 1 \leqslant i \leqslant M, \quad 1 \leqslant t \leqslant T(i) - 1\}. \tag{27}$$

If $T(i) = 1$ for any i, we delete the corresponding set in (25).

Now let the $\{\lambda^{ti}, 1 \leqslant i \leqslant M, 1 \leqslant t \leqslant T(i) - 1\}$ be renumbered $\{\lambda^t, 1 \leqslant t \leqslant T - 1\}$ with

$$0 < \lambda^t < \lambda^{t+1}, \quad 1 \leqslant t \leqslant T - 1.$$

Define the following sets $\{\Theta^{ti}\}, 1 \leqslant i \leqslant M, 1 \leqslant t \leqslant T - 1$:

$$\Theta^{ti} = \left\{ \begin{array}{ll} \{\theta^{r+1,i} & \text{if } \lambda^{ri} < \lambda^t < \lambda^{r+1,i} \text{ for some } r \\ \{\theta^{ri}, \theta^{r+1,i}\} & \text{if } \lambda^{ri} = \lambda^t \text{ for some } r \end{array} \right\}.$$

Then, the extreme points $f_E(Y^*)$ of $f(Y^*)$ are given by the non-dominated vectors in

$$\sum_{i=1}^{M} \sum_{t=1}^{T} \oplus \Theta^{ti},$$

where $\oplus$ means 'sum-set' addition (see [16], where an algorithm for finding the non-dominated vectors is given).

$f(Y^*)$ is then the piecewise linear set formed by joining up adjacent pairs of points of $f_E(Y^*)$.

Once $f(Y^*)$ has been found, the analysis of Section 3.1 and the approximation of Section 3.2, may be used.

4. Summary and comments

In this paper we have considered the problem of finding the efficient solution set, or an ϵ-efficient solution set, when the set of actions, X, may be expressed as the intersection of two other sets, $\{Y, Z\}$, each of which may be defined explicitly or implicitly. The central issue is one of whether or not this decomposition may be used to simplify the efficient solution set, or ϵ-efficient solution set, determination.

In Section 2 conditions are given to have $X^* = Z \cap Y^*$ and for $Z \cap Y^*$ to be a 0-efficient solution set for X. For situations where $\{X, Y, f(X), f(Y)\}$ are listed explicitly, a computational cost analysis is given to determine whether it is better to treat X directly and find $f(X^*)$ or to find $f(Y^*)$ first, and then find $f(Z) \cap f(Y^*)$. It is almost certain that it is better to take the former approach. If however, $\{X, Y, Z\}$ are given implicitly by constraints, then it is possible that Y has a special structure which facilitates the determination of $f(Y^*)$, which structure disappears as soon as the Z constraints are added. An illustration is given where this may arise, and the framework of an approach for finding $f(Y^*)$, and then $f(X^*)$, in the linear case for $K = 2$ is given.

In Section 3 an approach to the determination of whether, given an acceptable $\epsilon \in R^K_+$, $Z \cap Y^*$ is an ϵ-efficient solution set, is given for the linear case with $K = 2$. This is supplemented by the possibility of using approximations for $f(Y^*)$.

Finally, the ϵ-efficiency approach is illustrated with a problem in which $f(Y^*)$ is fairly readily obtained.

The general problem of determining whether or not a given ϵ leads to $Z \cap Y^*$ being an ϵ-efficient solution set is left as an open one. It is clearly a difficult problem and the computational effort involved may not be worthwhile. Nontheless, some extension to cases beyond the special ones considered in this paper may be possible.

For scalar optimisation problems considerable interest has been shown in the development of heuristics, whose purpose is to find acceptable solutions at acceptable computational effort levels. For multiple-objective problems, the need for heuristics is even greater, for, not only do we have the computational effort needed to find a single solution, but we have a great multiplicity of solutions in my cases. It is possible that, by looking at problem structures which facilitate such analysis, we may be able to develop heuristics for the multiple objective case. The approach of this paper is one such approach. If we find $f(Y^*)$ and then see how good $f(Z \cap Y^*)$ is, we may find that an acceptable ϵ exists. If this fails, all that is lost is the computational effort involved in this phase. At the very least we know that $Z \cap Y^* \subset X^*$, and hence some efficient solutions may be generated relatively easily.

An illustration of a special assignment problem is also given.

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References

1. Hwang, C.L., Masud, A.S.M., Paidy, S.R. and Yoon, K. (1979), *Multiple Objective Decision Making-Methods and Applications*. Springer Verlag, Berlin.
2. Zeleny, M. (1982), *Multiple Criteria Decision Making*. McGraw-Hill, New York.
3. Yu, P.L. (1985), *Multiple Criteria Decision Making.* Plenum Press, New York.
4. White, D.J. (1982), *Optimality and Efficiency*. John Wiley & Sons, Chichester.
5. White, D.J. (1990), A Bibliography on the Applications of Mathematical Programming Multiple-Objective Methods, *Journal of the Operational Research Society* 41, 669–691.

6. Benson, H.P. (1995), A Geometrical Analysis of the Efficient Outcome Set in Multiple Objective Convex Programs with Linear Criterion Functions, *Journal of Global Optimization* 6, 231–251.
7. White, D.J. (1991), A Characterisation of the Feasible Set of Objective Function Vectors in Linear Multiple Objective Problems, *European Journal of Operational Research* 52, 261–266.
8. Dauer, J.P. and Fosnaugh, T.A. (1995), Optimization over the Efficient Set, *Journal of Global Optimization* 7, 261–277.
9. Benson, H.P. (1994), Optimizing over the Efficient Set: Four Special Cases, *Journal of Optimization Theory and Applications* 80, 3–18.
10. Benson, H.P. (1991), An All-Linear Programming Relaxation Algorithm for Optimizing over the Efficient Set, *Journal of Global Optimization* 1, 83–104.
11. Philip, J. (1972), Algorithms for the Vector Maximization Problem, *Mathematical Programming* 2, 207–229.
12. White, D.J. (1996), The Maximization of a Function over the Efficient Set via a Penalty Function Approach, *European Journal of Operational Research* 94, 143–153.
13. Warburton, A. (1987), Approximation of Pareto Optima in Multiple-Objective, Shortest Path Problems, *Operations Research* 35, 70–79.
14. Kung, H.T., Lucio, f. and Preparata, F.P. (1975), On finding the Maxima of a Set of Vectors, *Journal of the Association for Computing Machinery* 22, 469–476.
15. White, D.J. (1984), A Special Multi-Objective Assignment Problem, *Journal of the Operational Research Society* 35, 759–767.
16. White, D.J. Generalised Efficient Solutions for Sums of Sets, *Operations Research* 28, 844–846, 1980.